Cluster Vertex Deletion on Chordal Graphs

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In the *cluster vertex deletion* problem, we are given a vertex-weighted graph and asked to find a maximum-weight set S of vertices so that $G[S]$ is a cluster graph (a graph whose components are all cliques). We write $\psi(G)$ for the weight of optimal solutions. We present a polynomial-time algorithm for this problem on chordal graphs, resolving an open question posed in different contexts by Cao et al. [2], Aprile et al. [1], and Hsieh et al. [3]. We use dynamic programming over clique trees and reduce the computation of the optimal subproblem to the minimization of a submodular set function.

Theorem 1. *There is a polynomial-time algorithm for cluster vertex deletion on chordal graphs.*

A chordal graph G has only linearly many maximal cliques, and these cliques can be arranged as a tree T such that, for every vertex $v \in V(G)$, the nodes of T corresponding to maximal cliques containing v form a subtree. For each node K (i.e., maximal clique of G) of the rooted clique tree T, we consider the subproblem on the subgraph G_K induced by the vertices that belong *only* to maximal cliques in the subtree rooted at K; vertices that also appear in the parent of K (if any) are excluded. If an optimal solution to this subproblem does not intersect K, it can be obtained by combining optimal solutions of the subproblems associated with the children of K. Consequently, the main difficulty is to handle optimal solutions that intersect K.

Any such solution can be readily reconstructed from the dynamic programming table once we know the unique cluster C that intersects K. This cluster must be a subset of some maximal clique Q in the subtree of K. We "guess" such a Q and a vertex $v \in K \cap Q$, and then choose the remaining vertices of C as a set $X \subseteq Q \setminus \{v\}$ that maximizes $\psi(G_K - N[X \cup \{v\}]) + w(X \cup \{v\})$: this objective is exactly the contribution of the cluster $C = X \cup \{v\}$ plus the best solution in the remaining components after deleting the cluster and its neighbors. The key structural ingredient is that, for fixed K, Q, and v, the objective function on X is supermodular: a function $f : 2^U \to \mathbb{Q}$ is *supermodular* if $f(X \cup Y) + f(X \cap Y) \geqslant f(X) + f(Y)$ for all $X, Y \subseteq U$. Thus, we can maximize it in polynomial time. The value of $\psi(G_K - N[X \cup \{v\}])$ is obtained from the dynamic programming table, since $G_K - N[X \cup \{v\}]$ decomposes into subgraphs corresponding to subtrees in the clique tree. Combining these cases for all nodes K yields the claimed algorithm.

We will prove the underlying structural statement in a more general form that might be of independent interest. Fix a clique K of a weighted chordal graph H. For any $X \subseteq K$, let $g(X)$ be the weight of a maximum-weight solution of H in which X is a cluster.

Theorem 2. *The function $g : 2^K \to \mathbb{Q}$ is supermodular.*

References

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